An Infinitesimal Analysis of the Stop-Loss-Start-Gain Strategy

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Abstract

The paradox of the Stop-Loss-Start-Gain trading strategy is resolved by showing that along the hyperfinite time-line the strategy incurs infinitesimal losses summing up to a non-infinitesimal amount. As a consequence the Black-Scholes formula is derived using only hyperreal arithmetic and Riemann sum, probably the most elementary derivation thus far.

We also discuss variable interest rate and the removal of arbitrage-free assumption in our setting.

1. BACKGROUND AND NOTATION

The Stop-Loss-Start-Gain (SLSG) is roughly the following strategy: fix a price level \(K\), if the stock price \(S_t\) at time \(t\) rises from \(< K\) to \(\geq K\), then buy 1 share immediately (SG); if \(S_t\) drops from \(\geq K\) to \(< K\), then sell immediately the share already hold, if any (SL); do nothing in other cases.

Assuming the Black-Scholes economic assumptions such as: no price spread between buy and sell, no restriction on short, trade takes place continuously in time, ... etc, the SLSG produces a portfolio which apparently needs no cost to run, yet, after SG, has a value \((S_T - K)^+\) at any time \(T\), thus duplicating the value of an European call option of strike price \(K\) expiring at \(T\). Hence there is an apparent contradiction to the Black-Scholes formula which gives a positive value to the European call option. This “paradox” was formally resolved by Carr and Jarrow in [2] by using the local time of Brownian paths. Study of SLSG in other settings can be found in [4] and [7]. The present article is inspired by M. Rubinstein’s essay [8] (under the name Seidenbverg). We use methods from nonstandard analysis to give a direct but rigorous resolution. In fact we derive the Black-Scholes formula using only hyperreal arithmetic and Riemann sum. Instead of using the infinite crossing property of Brownian paths, our approach here is based on a completely different philosophy. We use a hyperfinite discrete time-line. (Hyperfinite means infinite but

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finite in the sense of nonstandard analysis.) An infinitesimal loss is incurred whenever SL is triggered by the discrete movement of $S$ from $S_t \geq K$ to $K > S_{t+\Delta t}$. Then we show that such losses add up to the Black-Scholes price. Because of the naturalness of our approach, it lends support to the idea that in a fundamental way trades take place not continuously but discretely along a hyperfinite time-line.

In contrast to the nonstandard derivation of the Black-Scholes formula in either [3] or [6], our technique here is much more elementary: only arithmetic on the hyperreal numbers and basic Riemann sum suffice, neither the Loeb theory nor $\omega_1$-saturation principle is ever needed.

A brief reminder of some terminology from nonstandard analysis: we call infinite elements in the set $^*\mathbb{N}$ of nonstandard natural numbers hyperfinite; a set counted internally by a hyperfinite number is also called hyperfinite; given $r, s \in ^*\mathbb{R}$, hyperreal numbers, if $|r - s| < \rho$ for all $\rho \in \mathbb{R}^+$, we write $r \approx s$ (infinitely close); $r$ is called infinitesimal when $r \approx 0$; a finite element $r$ of $^*\mathbb{R}$ ($r < \infty$) is one with $|r| < n$ for some $n \in \mathbb{N}$; such $r$ is $\approx s$ for a unique $s \in \mathbb{R}$ (the standard part, in symbol: $s = \circ r$); write $r \approx \infty$ when $r$ is infinite. Complete background material can be found in [1],[5] or [6].

The present time is 0 and without loss of generality we take the terminal time to be 1. The hyperfinite time-line is defined as

$$T = \{0, \Delta t, 2\Delta t, \ldots, 1\},$$

where $\Delta t = 1/N$ and $N$ is a hyperfinite odd number.

Let $\mu, \sigma$ denote respectively the mean stock return rate (risk premium) and the positive non-infinitesimal volatility of the stock return rate. Let $r$ denote the riskless interest rate. $K$ denotes the critical price level set at time 1 which, when discounted from time 1, becomes $Ke^{r(t-1)}$ at time $t$. We emphasize that $\sigma, \mu, r, K$ are finite nonnegative constants.

With this set-up, at time $t + \Delta t$, SG is triggered when $S_t < Ke^{r(t-1)} \leq S_{t+\Delta t}$ and SL is triggered when

$$(SL) \quad S_t \geq Ke^{r(t-1)} > S_{t+\Delta t}.$$

We will concentrate mainly on SL, since this is when a cost is incurred.

By using $S_0$ as the numeraire, we can assume that $S_0 = 1$. 


By adding the constant \((1 - Ke^{-r})^+\) to \(S_0 = 1\), we can further assume that \(Ke^{-r} \geq 1\), i.e. \(K \geq r\).

Our model for the (risk premium-) discounted stock price \(S_t e^{\mu t}\) is given by a hyperfinite version of the Cox-Ross-Rubinstein centered binary tree in which the up ratio is \(u = e^{\sigma \sqrt{\Delta t}}\) and the down ratio \(d = e^{-\sigma \sqrt{\Delta t}}\), i.e. \(ud = 1\). Therefore we take as sample space \(\Omega = \{-1, +1\}^T\) and for \(\omega \in \Omega\)

\[
S_{t+\Delta t}(\omega)e^{-\mu(t+\Delta t)} = S_t(\omega)e^{-\mu t}e^{r}\Delta t} = S_t(\omega)e^{-\mu t}e^{k\sigma \sqrt{\Delta t}},
\]

i.e. \(S_{t+\Delta t}(\omega) = S_t(\omega)e^{\omega \sigma \sqrt{\Delta t} + \mu \Delta t}\). We let constant \(p\) denote the up transitional probability at each time \(t\). Assume that the discounted stock price obeys arbitrage-free at every time \(t\), we must have

\[
E_t[\{S_t \geq Ke^{(t-1)\mu}\} \{S_t \geq S_{t+\Delta t}\}] = S_t e^{-\mu t} e^{\rho \Delta t},
\]

i.e. \(pu + (1-p)d = e^{r \Delta t}\), or

\[
p = \frac{e^{r \Delta t} - d}{u - d} = \frac{e^{r \Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}.
\]

We remark that since \(\sigma > 0\) is finite non-infinitesimal and \(\Delta t \approx 0\), it follows that \(p \approx \frac{1}{2}\).

In this model, the total expected loss discounted back to the present is:

\[
\sum_{0 \leq t < 1} e^{-r(t+\Delta t)} E_t \left[ \{S_t \geq Ke^{(t-1)\mu}\} \{S_t \geq S_{t+\Delta t}\} \right].
\]

(Note that the infinitesimal loss \(S_t - S_{t+\Delta t}\) actually occurs at time \(t + \Delta t\).)

In the next section, we will perform some explicit hyperreal computations and obtain the Black-Scholes formula from (1). In Section 3, we derive a general Black-Scholes price under arbitrage-free assumption and a variable interest rate. In Section 4, we remove the arbitrage-free assumption from our setting and analyze the cost of SLSG corresponding to all possible constant transitional probability.

2. Summing up the infinitesimal losses

Now we analyze the expected total loss (1) carefully by transforming it into a sequence of expressions (2) – (13), where \((i) = \alpha (i - 1) + \beta\) for some \(\alpha \approx 1\) and \(\beta \approx 0\). Therefore if \((i)\) is finite then \((i) \approx (i - 1)\); moreover the final one (13) will be equal to the Black-Scholes price.
When SL is triggered along the path \( \omega \) at time \( t + \Delta t \), \( t \) must be of the form \( 2n\Delta t \) for some \( n \in \mathbb{N} \), and there is an \( m = m_n \), the least integer satisfying 
\[
e^{-2m(\sigma \sqrt{\Delta t} + \mu \Delta t)} \geq Ke^{r(t-1)} \quad \text{(note both sides are infinitely close to each other)}, \] such that \( \{\omega_s, 0 \leq s < t\} \) consists of \( (n + m) \) many +1’s and \( (n - m) \) many -1’s with \( \omega_t = -1 \). Therefore the number of SL triggering paths at such time is \( \frac{1}{2} \binom{2n}{n+m} \). The infinitesimal loss on such path \( \omega \) at time \( t + \Delta t \) is

\[
S_t(\omega) - S_{t+\Delta t}(\omega) = e^{2m(\sigma \sqrt{\Delta t} + \mu \Delta t)}(1 - e^{-\sigma \sqrt{\Delta t} + \mu \Delta t}).
\]

So the expected total loss (1) equals

\[
(2) \quad \sum_{n=0}^{N-1} e^{-r(2n+1)\Delta t} e^{2m(\sigma \sqrt{\Delta t} + \mu \Delta t)}(1 - e^{-\sigma \sqrt{\Delta t} + \mu \Delta t}) \frac{1}{2} \binom{2n}{n+m} p^{n+m}(1-p)^{n-m}.
\]

By the choice of \( m \) and \( 1 - e^{-\sigma \sqrt{\Delta t} + \mu \Delta t} = \sigma \sqrt{\Delta t} + O(\Delta t) \) (in the standard big oh sense) we transform (2) into

\[
(3) \quad \sum_{n=0}^{N-1} e^{-r(2n)\Delta t} (Ke^{r(2n-1)\Delta t})(\sigma \sqrt{\Delta t}) \frac{1}{2} \binom{2n}{n+m} p^{n+m}(1-p)^{n-m}.
\]

Write \( k = \left[ \frac{\ln K + r(2n \Delta t - 1)}{2(\sigma \sqrt{\Delta t} + \mu \Delta t)} \right] \sqrt{\Delta t} \), i.e. \( m = k \frac{\sqrt{\Delta t}}{2(\sigma \sqrt{\Delta t} + \mu \Delta t)} \). Notice that \( k \) is finite and nonnegative. Choose some hyperfinite \( M \) such that \( M \Delta t \approx 0 \) but \( M \sqrt{\Delta t} \approx \infty \). (For example, \( M = N^{2/3} \).) Notice that \( m = k \frac{\sqrt{n \Delta t}}{n \sqrt{\Delta t}} \approx 0 \) whenever \( n \geq M \). Then the (3) is transformed into

\[
(4) \quad \sum_{n=M}^{N-1} Ke^{-r(\sigma \sqrt{\Delta t})} \frac{1}{2} \binom{2n}{n+m} p^{n+m}(1-p)^{n-m}
\]

i.e.

\[
(5) \quad \frac{K\sigma}{2e^r} \sum_{n=M}^{N-1} \left( \frac{2n}{n+m} \right) p^n(1-p)^n \left( \frac{p}{1-p} \right)^m \sqrt{\Delta t}.
\]

We apply the following form of the Stirling’s formula

\[
h! = \sqrt{2\pi h} \left( \frac{h}{e} \right)^h e^{\frac{1}{2}} \quad \text{for some} \quad 0 < \epsilon < \frac{1}{12}
\]

and obtain

\[
\binom{2n}{n+m} = \frac{2^{2n}}{\sqrt{\pi n}} \left( 1 - \left( \frac{m}{n} \right)^2 \right)^{-\frac{1}{2}} \left( \frac{(n+m)^{n+m}(n-m)^{n-m}}{n^{2n} \lambda_{n,m}} \right) \lambda_{n,m}
\]
for some $\lambda_{n,m}$ which is $\approx 1$ whenever $n, n - m$ are infinite — in particular when $n \geq M$. For such $n$, since $\frac{m}{n} \approx 0$, 
\[
(1 - \left(\frac{m}{n}\right)^2)^{-\frac{1}{2}} \approx 1
\]
and
\[
(n + m)^n(n - m)^{-n} = \left(1 - \frac{m}{n + m}\right)^{n+m} \left(1 - \frac{m}{n - m}\right)^{-n-m} < e^{-m}e^m = 1.
\]

Therefore (5) is transformed into
\[
\begin{align*}
K\sigma & \sum_{n=M}^{N-1} \frac{1}{\sqrt{\pi n\Delta t}} \left(\frac{n^{2n}}{(n + m)^{n+m}(n - m)^{n-m}}\right) e^{2^n p^n (1 - p)^n} \left(\frac{p}{1 - p}\right)^m \Delta t.
\end{align*}
\]

A little computation shows that
\[
N \ln \left(4p(1-p)\right) = N \ln \left(4\left(\frac{e^{r\Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t} - e^{r\Delta t}}}ight)\left(\frac{e^{\sigma \sqrt{\Delta t} - e^{r\Delta t}}}{e^{\sigma \sqrt{\Delta t} - e^{r\Delta t}}}ight)\right) = -\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)^2 + O(\Delta t),
\]
so
\[
2^n p^n (1 - p)^n \approx e^{-\left(\frac{p}{1 - p}\right)^m}.
\]

Another computation shows that
\[
\left(\frac{p}{1 - p}\right)^m = \left(\frac{e^{r\Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t} - e^{r\Delta t}}}\right)^m = \left(1 + \left(\frac{2r}{\sigma} - \sigma\right)^{\sqrt{\Delta t}} + O(\Delta t)\right)^m \approx e^{\left(\frac{2r}{\sigma} - \sigma\right)^{\sqrt{\Delta t}}}.
\]

Using these and (*), we transform (6) into
\[
\begin{align*}
K\sigma & \sum_{n=M}^{N-1} \frac{1}{\sqrt{\pi n\Delta t}} \left(\frac{n^{2n}}{(n + m)^{n+m}(n - m)^{n-m}}\right) e^{-\left(\frac{p}{1 - p}\right)^m} e^{-\left(\frac{2r}{\sigma} - \sigma\right)^{\sqrt{\Delta t}}} \Delta t.
\end{align*}
\]

We have
\[
\frac{n^{2n}}{(n + m)^{n+m}(n - m)^{n-m}} = \left(1 - \left(\frac{m}{n}\right)^2\right)^{n} \left(\frac{n - m}{n + m}\right)^{m},
\]
also for $n\Delta t \neq 0$,
\[
\left(1 - \left(\frac{m}{n}\right)^2\right)^{n} \approx e^{\frac{m^2}{n}} = e^{\frac{m^2}{n\Delta t}}
\]
and
\[
\left(\frac{n - m}{n + m}\right)^{m} = \left(1 - \frac{2m}{n + m}\right)^{m} = \left(1 - \frac{2k}{n\Delta t\sqrt{\sigma} + k}\right)^{m} \approx e^{-\frac{2k^2}{n\Delta t}},
\]

so
therefore
\[
\frac{n^{2n}}{(n + m)^{n+m}(n - m)^{n-m}} \approx e^{-\frac{k^2}{2\Delta t}} \quad \text{when} \quad n\Delta t \neq 0.
\]
Hence from this and (*), we transform (7) into
\[
K\sigma \frac{N}{2e^r} \sum_{n=M}^{N-1} \frac{1}{\sqrt{\pi n\Delta t}} e^{-\frac{k^2}{2\Delta t} - (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})^2 n\Delta t + (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})k \Delta t}.
\]
Write
\[
c_t = \frac{\ln K + r(t - 1)}{2\sigma},
\]
note that
\[
k = k_n = \left[ \ln K + r(2n\Delta t - 1) \right] \frac{1}{2(\sigma\sqrt{\Delta t} + \mu \Delta t)} \sqrt{\Delta t} \approx c_{2n}\Delta t
\]
and \(\mu\) vanishes from \(c_t\). We transform (8) into
\[
K\sigma \frac{N}{2e^r} \sum_{n=M}^{N-1} \frac{1}{\sqrt{\pi n\Delta t}} e^{-\frac{c_{2n}\Delta t^2}{2\Delta t} - (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})^2 n\Delta t + (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})c_t \Delta t}.
\]
When viewed as a hyperfinite Riemann sum (which is finite), (9) is infinitely close to the integral
\[
K\sigma \frac{1}{2e^r} \int_0^1 \frac{1}{\sqrt{\pi t}} e^{-\frac{c_{2n}\Delta t^2}{2\Delta t} - (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})^2 t + (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})c_t} dt
\]
or equivalently
\[
K\sigma \frac{1}{2e^r} \int_0^1 \frac{1}{\sqrt{2\pi t}} e^{-\frac{c_{2n}\Delta t^2}{2\Delta t} - (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})^2 t + (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})c_t} dt.
\]
Clearly this integral is positive and non-infinitesimal, hence we have already resolved the SLSG paradox by proving that it has a positive non-infinitesimal cost.

We now continue to show (11) gives the Black-Scholes price.

Using the definition of \(c_t\) and completing the square, we transform (11) into
\[
K\sigma \frac{1}{2e^r} \int_0^1 \frac{1}{\sqrt{2\pi t}} e^{-\frac{c_{2n}\Delta t^2}{2\Delta t} - (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})^2 t + (\frac{k}{\sigma} - \frac{1}{\sqrt{2\pi}})c_t} dt.
\]
By changing the variable with \(x = \sigma\sqrt{t}\), the above equals
\[
Ke^{-r} \int_0^{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \ln K}{\sigma} \right)^2} dx.
\]
Therefore the the total expected loss (1) is infinitely close to this finite integral.

Moreover, by differentiating with respect to \(\sigma\) the Black-Scholes formula
\[
N\left( \frac{r - \ln K}{\sigma} + \frac{\sigma}{2} \right) - Ke^{-r}N\left( \frac{r - \ln K}{\sigma} - \frac{\sigma}{2} \right),
\]
where
\[
N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du,
\]
then integrating back using integrating constant 0 (since when $\sigma = 0$, the Black-Scholes price is 0), we get precisely (13).

Therefore (13) equals the Black-Scholes formula, i.e.

**Theorem 1.** Under the arbitrage-free assumption, the Stop-Loss-Start-Gain strategy has a cost given by the Black-Scholes formula. □

3. **Under variable deterministic interest rate**

Results in Section 2 can be modified in the case when the riskless interest rate is not necessarily constant but a deterministic function $r_t$. It is reasonable to assume that $r_t$ is piecewise $S$-continuous in the sense that for $t, t' \in T$,

$$r_t \approx r_{t'} \quad \text{whenever} \quad t \approx t' \quad \text{except for finitely many} \quad t.$$

Moreover $r_t \geq 0$ and is finite. In particular $r_t, t \in T$ has a finite bound. Furthermore, using the standard part, $\rho(\omega t) = \omega r_t$ defines a standard piecewise continuous function $\rho : [0, 1] \to \mathbb{R}$ and

$$\sum_{s \leq u \leq t} r_u \Delta t \approx \int_s^t \rho_u du \quad \text{for all} \quad s \leq t \quad \text{in} \quad T.$$

(Conversely, any standard piecewise continuous $\rho : [0, 1] \to \mathbb{R}$ has a corresponding piecewise $S$-continuous $r$ satisfying the above.)

Then (SL) becomes:

(SL')

$$S_t \geq Ke^{-\sum_{t < s \leq t} r_s \Delta t} > S_{t+\Delta t}.$$

and (1) becomes

(14) \[ \sum_{0 \leq t < 1} e^{-\sum_{t < s \leq t} r_s \Delta t} E \left[ 1_{\{S_t \geq Ke^{-\sum_{t < s \leq t} r_s \Delta t} > S_{t+\Delta t}\}}(S_t - S_{t+\Delta t}) \right]. \]

The other modifications needed are the transitional probability

$$\tilde{p}_t = \frac{e^{\rho_1 \Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}, \quad t = n\Delta t,$$

and

$$\tilde{k}_t = \left[ \ln K - \sum_{t < s \leq t} r_s \Delta t \right] \frac{1}{2(\sigma \sqrt{\Delta t} + \mu \Delta t)} \sqrt{\Delta t}, \quad t = n\Delta t,$$

and

$$\tilde{c}_t = \frac{1}{2\sigma} \left( \ln K - \int_1^t \rho_u du \right).$$

From the Black-Scholes price (11) we have
**Theorem 2.** Suppose the riskless interest rate is given by a deterministic piecewise continuous function $\rho_t$, then under the arbitrage-free assumption, the Black-Scholes price is

$$K\sigma^2 e^{R} \int_0^1 \frac{1}{\sqrt{2\pi t}} e^{-\frac{r^2}{2} - (\frac{1}{2} - \rho_t)^2 + (\frac{2}{\sigma^2} - \rho_t)\tilde{c}_t} dt,$$

where $R = \int_0^1 \rho_s ds$ and $\tilde{c}_t = \frac{1}{2\sigma} \left( \ln K - \int_0^1 \rho_s ds \right)$.

4. **The cost of SLSG in an arbitrage environment**

We now consider the same model in Section 1 but the arbitrage-free assumption is removed. That is, $p$ is not necessarily given by $e^{r\Delta t - d}$.

We do this for this simple case when $K = 1$ and $r = 0$. As a result $k = m = 0$ and the cost (6) becomes

$$K\sigma^2 e^{R} \sum_{n=M}^{N-1} \frac{1}{\sqrt{\pi n\Delta t}} 2^{2n} p^n (1 - p)^n \Delta t.$$

We would like to discuss the cost over all possible values of $p$.

First note that (15) equals 0 when $p = 0$ or $p = 1$. Now we exclude these cases and write $p = \frac{1}{2} + \epsilon$, where $-\frac{1}{2} < \epsilon < \frac{1}{2}$.

Also $4p(1-p) = 4(\frac{1}{2} + \epsilon)(\frac{1}{2} - \epsilon) = 1 - (2\epsilon)^2$ and (15) equals

$$K\sigma^2 e^{R} \sum_{n=M}^{N-1} \frac{1}{\sqrt{\pi n\Delta t}} (1 - (2\epsilon)^2)^n \Delta t,$$

and this hyperfinite Riemann sum is infinitely close to

$$K\sigma^2 e^{R} \int_0^\frac{1}{2} \frac{1}{\sqrt{\pi}} (1 - (2\epsilon)^2)^N \Delta t.$$

Upon a change of variable, we get

$$K\sigma^2 e^{R} \int_0^\frac{1}{\sqrt{\pi}} (1 - (2\epsilon)^2)^N \Delta \sqrt{\frac{1}{\pi}} dt.$$

Write $\alpha = -\ln \sqrt{1 - (2\epsilon)^2} > 0$. Then (18) equals

$$K\sigma^2 e^{R} \int_0^{\alpha} \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}} N \Delta \alpha dt.$$

Change variable again, we get

$$K\sigma^2 e^{R} \int_0^{\alpha} \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}} N \Delta \alpha dx.$$
Theorem 3. Let \( D = N \sum_{n=1}^{\infty} \frac{(2p-1)^{2n}}{n} \). Then when the arbitrage-free assumption is removed, SLSG has the following possible costs:

(i) if \( p = 0 \) or \( p = 1 \) the cost is 0;
(ii) if \( D \approx \infty \), the cost is infinitesimal;
(iii) if \( D \approx 0 \), the cost is \( \approx K\sigma \sqrt{2\pi e D} \);
(iv) if \( D \) is finite but \( \neq 0 \), the cost is positive and < \( \frac{K\sigma}{2\sqrt{D}e^D} \).

Proof. (i) was already discussed above.

\[
\text{In (20), } N\alpha = -N \ln \sqrt{1-(2\epsilon)^2} = N \sum_{n=1}^{\infty} \frac{(2\epsilon)^{2n}}{2n} = N \sum_{n=1}^{\infty} \frac{(2p-1)^{2n}}{2n} = \frac{D}{2},
\]

so the cost is infinitely close to

\[
\frac{K\sigma}{\sqrt{D}e^D} \int_0^{\sqrt{D}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,
\]

and conclusions (ii) - (iv) follow from it. \(\Box\)

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