PDE Methods for Option Pricing under Jump Diffusion Processes

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Summary

• Merton jump diffusion

• American options

• Levy Processes - Variance Gamma and CGMY

• Asian Options under Jump Diffusions

• Conclusions
Merton’s Jump-Diffusion model

Merton’s model consists of a jump term added to geometric Brownian motion i.e.

\[ \frac{dS_t}{S_t} = (\mu - \lambda \kappa) dt + \sigma dW_t + dJ_t \]

where \( J_t \) is a compound Poisson process with rate \( \lambda \) i.e.

\[ J_t = \sum_{j=1}^{N_t} (Y_j - 1), \quad \text{where } E[N_t] = \lambda t \text{ and } P\{N_t = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \]

If the \( j^{th} \) jump occurs at time \( t \) and \( S_{t-} \) is the asset price immediately before the jump, then \( S_t = Y_j S_{t-} \). The compensation constant \( \kappa = E[Y - 1] \).
Merton derives an analytic solution for a call option when the $Y$ are lognormally distributed

$$\ln Y \sim N(a, b^2), \text{ i.e. } f(y) = \frac{1}{y\sqrt{2\pi}b} \exp\left\{ -\frac{(\ln y - a)^2}{2b^2} \right\} \Rightarrow \kappa = e^{(a+0.5b^2)} - 1$$

The solution to the jump-diffusion sde is

$$S_t = S_0e^{(\mu - \lambda \kappa - 0.5\sigma^2)t + \sigma W_t} \prod_{j=1}^{N_t} Y_j$$

so multiplicative jumps can be seen to be a natural extension of GBM.
Levy Processes

This jump-diffusion model is an example of a more general class of exponential Levy models

\[ S_t = S_0 e^{X_t} \]

where \( X_t \) is a Levy process consisting of a drift term, a Brownian motion term and a superposition of Poisson processes with various jump sizes and allowing for infinite jump rates or activities (of vanishingly small jumps). The Merton model has a Levy density

\[ \nu(x) = \frac{\lambda}{\sqrt{2\pi}b} \exp \left\{ -\frac{(x - a)^2}{2b^2} \right\}, \text{ where } x = \ln y \]

and is a finite activity process (\( \int_{-\infty}^{\infty} \nu(x) dx = \lambda \)).
Variance Gamma

Variance Gamma is a Levy process and is widely used as a share price model; it has infinite but "relatively low" activity of small jumps

\[ \nu(x) = \frac{\exp\{-\lambda_p x\}}{cx} 1_{x>0} + \frac{\exp\{-\lambda_n |x|\}}{c|x|} 1_{x<0}, \text{ where } x = \ln y \]
CGMY is another Levy process and is widely used as a share price model; it is identical to Variance Gamma if $Y = 0$

$$\nu(x) = C \frac{\exp \{-Mx\}}{x^{1+Y}} 1_{x>0} + C \frac{\exp \{-G|x|\}}{|x|^{1+Y}} 1_{x<0}, \text{ where } x = \ln y$$
Pricing PIDE

Under certain assumptions, the Option price $V$, when the underlying follows an exponential Levy process, can be shown to obey the following partial-integro-differential equation

$$-rac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V + \int_{-\infty}^{\infty} (V(Se^x) - V(S) - S(e^x - 1) \frac{\partial V}{\partial S}) \nu(x) dx$$

or

$$-rac{\partial V}{\partial t} = \mathcal{L}V + \mathcal{I}(V)$$

or inequality in the case of American Options. A flexible approach to pricing is to solve a finite difference approximation to this PIDE/PIDI.
The PIDE can be integrated between two time levels at

\[
\int_{t^{m-1}}^{t^m} -\frac{\partial V}{\partial t} \, dt = V(S_j, t^{m-1}) - V(S_j, t^m) = \int_{t^{m-1}}^{t^m} (\mathcal{L} V(S_j, t) + \mathcal{I}(V(S, t))) \, dt
\]

and using \(\theta\) method time quadrature leads to the finite difference approximation

\[
V_j^{m-1} - V_j^m = \Delta t \left[ \theta (\mathcal{L}_{\Delta S} V_j^{m-1} + \mathcal{I}_j(V^{m-1})) + (1 - \theta) (\mathcal{L}_{\Delta S} V_j^m + \mathcal{I}_j(V^m)) \right].
\]

where \(\mathcal{L}_{\Delta S}\) and \(\mathcal{I}_j(V^m)\) are 2\(^{nd}\) order approximations to the continuous operators, and \(V_j^m\) are discrete prices. A uniform spatial mesh in \(S\), or in \(u = \ln S\), would not distribute the error efficiently; a better choice is either to use a non-uniform meshes (Forsyth, Toivanen) or uniform meshes with coordinate stretching (this work).
The region of asset price interest is determined by the payoff; \( S = k \) for a vanilla call or put. In this case a simple analytic transformation \( S = a \sinh(\xi - L) + c \) can be used to refine the mesh around \( S = K \).

Choose the curve to pass through \((0, 0)\) \(\Rightarrow c = a \sinh L\) and the maximum stretch to be at \( S = K \) \(\Rightarrow c = K\).

\[
i.e. \quad S = \frac{K}{\sinh(L)} \sinh(\xi - L) + K.
\]

The parameter \( L \) controls the stretch. The transformed equations are more complex, however implementing the finite difference equations is straightforward (on a uniform mesh in the transformed variable).
$y = \sinh(x)$

Non-Uniform set of intervals in $y$

Uniform set of intervals in $x$
Comments on the convolution integral

- The integrand is localised by the exponential tapering of a Levy density and moreover is identically zero when the prices are locally linear.

- The main accuracy requirement is consequently around the strike region (or between the strike and the optimal exercise boundary). Local mesh refinement of this region will control errors both in the Black-Scholes operator and convolution quadrature.

- Using $V_{SS} = 0$ at $S = S_{max}$ as a mesh truncation condition enforces linearity in the prices beyond $S_{max}$. The coordinate stretching is also an efficient way to provide a locally linear region to the right of the strike (mesh size increases exponentially).
Evaluating the discrete convolution integral

\[ \mathcal{I}^j(V) = \int_{-\infty}^{\infty} \left( V(S_j e^x) - V_j - S_j (e^x - 1) \left[ \frac{\partial V}{\partial S} \right]_{S_j} \right) \nu(x) dx \]

Approximating this integral using quadrature appears to require interpolation for \( V(S_j e^x) \), however this can be avoided. This is easier to follow with a transformation back to the asset price jump \( y = e^x \) i.e.

\[ \mathcal{I}_j(V) = \int_0^\infty \left( V(S_j y) - V_j - S_j (y - 1) \left[ \frac{\partial V}{\partial S} \right]_{S_j} \right) \tilde{\nu}(y) dy \]

\[ = \int_0^\infty F_j(y) \tilde{\nu}(y) dy \]
Localisation

Let $\Omega = \{S_0, S_1, \ldots S_N = S_{max}\}$ be the finite difference mesh in $S$. Then the following localisation

$$ I_j \approx I_j^{loc} = \int_{0}^{y_j^{max}} F_j(y)\tilde{\nu}(y)dy \text{ where } y_j^{max} = S_N/S_j $$

is accurate providing that mesh truncation is sufficiently far from the strike.

Note that the effect of asset price jumps outside the mesh are ignored e.g. for $j = N$, $I_N(V) \approx \int_{0}^{1} F_N(y)\tilde{\nu}(y)dy$; this is consistent with imposing the truncation condition $V_{SS} = 0$, at $S = S_{max}$ since the integrand will then be zero for $S > S_{max}$. 
Mesh based quadrature

For a given localisation $y_j^{\max} = y_N = S_N/S_j$, construct a partition of all possible jumps $\{[y_0, y_1], \ldots, [y_p, y_{p+1}], \ldots, [y_{N-1}, y_N]\}$. Then

$$I_{j}^{\text{loc}} = \sum_{p=0}^{p=N-1} \int_{y_p}^{y_{p+1}} F_j(y) \tilde{\nu}(y) dy$$

Applying the trapezoidal rule to each interval,

$$I_{j}^{\text{loc}} \approx \hat{I}_{j}^{\text{loc}} = \sum_{p=0}^{p=N-1} \left( \frac{1}{2} (y_{p+1} - y_p) [F_j(y_{p+1}) \tilde{\nu}(y_{p+1}) + F_j(y_p) \tilde{\nu}(y_p)] \right)$$
Discrete densities

The localised integral term can be rewritten as

\[
\tilde{I}_j^{loc} = \frac{(y_N - y_{N-1})}{2} F_j(y_N) \tilde{\nu}(y_N) + \sum_{p=1}^{p=N-2} \frac{(y_{p+1} - y_{p-1})}{2} F_j(y_p) \tilde{\nu}(y_p) \\
+ \frac{(y_1 - y_0)}{2} F_j(y_0) \tilde{\nu}(y_0) \\
= \sum_{p=0}^{p=N} \nu_{jp} F_j(y_p)
\]

where \( F_j(y_p) = V_p - V_j - (y_p - 1) \left[ \frac{\partial V}{\partial S} \right]_{S_j} \) and the \( \nu_{jp} \) are discrete densities for a jump from \( S_j \) to \( S_p \).
Quadrature with coordinate transformations

1. The non-uniform mesh in $S$ creates a non-uniform set of quadrature intervals in $y = S/S_j$.

2. The quadrature sampling is most accurate in the region of the greatest transformation stretch (e.g. the strike $S = K$) i.e. where the integrand has its maximum variation.

3. Simpson’s rule (note that the quadrature rule is uniformly spaced in $\xi$) improves the discrete quadrature measures but computational tests show little effect on the option prices.
European Price Convergence - Merton

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Table 1: Numerical convergence comparison of uniform and stretched grids for an European Put with Merton jump diffusion. Prices at \( S=100 \), where \( K = 100, T = 0.25, r = 0.05, \sigma = 0.15, \lambda = 0.1, a = -0.9, b = 0.45 \). Exact value = 3.1490
American Options

The pricing PIDE is now replaced by an inequality since the hedged portfolio value can only have a return bounded above by the risk-free return i.e.

\[-\frac{\partial V}{\partial t} \geq \mathcal{L}V + \int_0^{\infty} (V(Sy) - V(S) - (y-1)S\frac{\partial V}{\partial S})\tilde{\nu}(y)dy = \mathcal{L}V + \mathcal{I}(V)\]

An American option cannot fall beneath its immediate payoff $g(S, t)$, thus

\[V \geq g\]

must also hold. These two conditions combine into a linear complementarity problem

\[(V - g)(\frac{\partial V}{\partial t} + \mathcal{L}V + \mathcal{I}(V)) = 0\]
Discrete Complementarity

The discrete linear complementarity problem is

\[(V_j^{m-1} - g(S_j, t^{m-1})) \times \]

\[
\left( \frac{V_j^{m-1} - V_j^m}{\Delta t} - \left[ \theta(\mathcal{L}_{\Delta S} V_j^{m-1} + \mathcal{I}_j(V_{m-1})) + (1 - \theta)(\mathcal{L}_{\Delta S} V_j^m + \mathcal{I}_j(V^m)) \right] \right) = 0
\]

Rannacher timestepping is effective in avoiding oscillations in Gamma (cf Giles et al).
## American Price Convergence - Merton

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Table 2: Numerical convergence comparison of uniform and stretched grids for an American Put with Merton jump diffusion. Prices at $S=100$, where $K = 100$, $T = 0.25$, $r = 0.05$, $\sigma = 0.15$, $\lambda = 0.1$, $a = -0.9$, $b = 0.45$. Exact value $\approx 3.2412$
Figure 1: Stretched grid Gamma for Merton jump diffusion, American Put
\textbf{Algebraic structure}

\begin{align*}
a_j V_{j-1}^{m-1} + b_j V_j^{m-1} + c_j V_{j+1}^{m-1} - \theta \Delta t \sum_{p=0}^{p=N} \nu_{jp} F_p &= \\
A_j V_{j-1}^m + B_j V_j^m + C_j V_{j+1}^m + (1 - \theta) \Delta t \sum_{p=0}^{p=N} \nu_{jp} F_p, \quad j = 0, \ldots N
\end{align*}

In matrix notation

\begin{equation}
(M + M_c) V^{m-1} = f
\end{equation}

See approaches in Forsyth et al and Cont and Voltchkova for fast techniques for dealing with the computational cost of the full matrix $M_c$. PSOR has been found to work satisfactorily for the implicit case and early exercise. Note that $M_c$ entries will decay exponentially fast away from the diagonal.
Infinite activity Levy densities

This approach needs to be modified for Variance Gamma and CGMY since
\( \tilde{\nu}(y_p) = \tilde{\nu}(y_p) \) is singular for \( p = j \) i.e. \( y = 1 \). The terms that have to be treated separately in \( \mathcal{L}_j \) are

\[
\sum_{p=j-1}^{p=j} \int_{y_p}^{y_{p+1}} F(S_j y) \tilde{\nu}(y) dy = \int_{y_{j-1}}^{y_{j+1}} \left( V(S_j y) - V(S_j) - S_j (y - 1) \frac{\partial V}{\partial S} \right) \tilde{\nu}(y) dy
\]

The approach described here follows Rama Cont (and also Peter Forsyth).

\[
y_{j-1} = \frac{S_{j-1}}{S_j} \quad y_j = 1 \quad y_{j+1} = \frac{S_{j+1}}{S_j}
\]
Effective volatility $\sigma_\epsilon$

\[ I_j^{sing} = \int_{y_{j-1}}^{y_{j+1}} \left( V(S_jy) - V(S_j) - S_j(y - 1) \frac{\partial V}{\partial S} \right) \tilde{\nu}(y) dy \]

Expand $V(S_jy)$ in a Taylor series about $S_j$ for small jumps $y - 1 = \epsilon \ll 1$

\[ V(S_jy) = V(S_j) + S_j(y - 1) \frac{\partial V}{\partial S} + \frac{1}{2} S_j^2 (y - 1)^2 \frac{\partial^2 V}{\partial S^2} + ... \]

Then $I_j^{sing} = \frac{1}{2} \sigma_\epsilon^2 S_j^2 \frac{\partial^2 V}{\partial S^2}$

where $\sigma_\epsilon^2 = \int_{y_{j-1}}^{y_{j+1}} (y - 1)^2 \tilde{\nu}(y) dy$. 

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Evaluating $\sigma_\epsilon^2$

$$\sigma_\epsilon^2 = \int_{x_{j-1}}^{x_{j+1}} (e^x - 1)^2 C \left[ \frac{e^{-Mx}}{x^{1+Y}} 1_{x>0} + \frac{e^{-G|x|}}{|x|^{1+Y}} 1_{x<0} \right] dx,$$ where $x = \ln(y)$.

\begin{align*}
x_{j-1} &= \ln\left(\frac{S_{j-1}}{S_j}\right) & x_j &= 0 & x_{j+1} &= \ln\left(\frac{S_{j+1}}{S_j}\right) \\
S_{j-1} &\leftarrow \Delta S_{j-1} \rightarrow S_j & S_j &\leftarrow \Delta S_j \rightarrow S_{j+1}
\end{align*}

Define $\Delta x^D_j = x_j - x_{j-1} = 0 - \ln\left(\frac{S_{j-1}}{S_j}\right) = \ln(1 + \frac{\Delta S_{j-1}}{S_{j-1}}) \approx \Delta S_{j-1}/S_j$

and $\Delta x^U_j = x_{j+1} - x_j = \ln\left(\frac{S_{j+1}}{S_j}\right) - 0 = \ln(1 + \frac{\Delta S_j}{S_j}) \approx \Delta S_j/S_j$
Then
\[ \sigma^2 = \int_0^\Delta x_j U e^{-Mx} \frac{(e^x - 1)^2}{x^{1+Y}} \, dx + \int_0^\Delta x_j D e^{-Gx} \frac{(e^x - 1)^2}{x^{1+Y}} \, dx \]

\[ = \int_0^\Delta x_j U x^{1-Y} e^{-Mx} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)^2 \, dx + \int_0^\Delta x_j D x^{1-Y} e^{-Gx} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)^2 \, dx \]

which for \( Y < 2 \), can be expressed in terms of incomplete Gamma functions, and converges quickly (3 terms needed).
### European Price Convergence - Variance Gamma

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Table 3: Numerical convergence of European Call Prices with Variance Gamma comparing different stretched grids with Rannacher timestepping. Prices linearly interpolated to $S=90$ where $K = 98$, $r = 0.0$, $T = 0.5$, $\nu = 0.1686$, $\lambda_n = 20.264$, $\lambda_p = 39.784$. Convergence rates not quoted as they were affected by the interpolation. Exact $\approx 0.61337$. 

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Figure 2: VG American Put showing lack of smooth pasting
### American Price Convergence - Variance Gamma

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Table 4: Numerical convergence comparison of uniform and stretched grids for an American Put under VG. Prices at S=100, where \( K = 100 \), \( T = 0.5 \), \( r = 0.05 \), \( \sigma = 0.0 \), \( \nu = 0.1686 \), \( \lambda_n = 20.264 \), \( \lambda_p = 39.784 \). Limiting value \( \approx 2.90347 \)
## European Price Convergence - CGMY

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Table 5: Comparison of stretched grid (10 : 1) with Wang, Wan and Forsyth (2006) for CGMY. European Put prices linearly interpolated to $S = 90$ where $K = 98$, $r = 0.06$, $\sigma = 0.0$, $C = 16.97$, $G = 7.08$, $M = 29.97$, $Y = 0.6442$. Exact value is 16.2126
# American Price Convergence - CGMY

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<td>9.2181</td>
<td>0.0004</td>
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<tr>
<td>257</td>
<td>9.2215</td>
<td>0.0030</td>
<td>9.2238</td>
<td>0.0053</td>
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<tr>
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<td>0.0030</td>
<td>9.2249</td>
<td>0.0064</td>
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Table 6: Comparison of stretched grid (10 : 1) with Wang, Wan and Forsyth (2006) for CGMY. American Put prices linearly interpolated to $S = 90$ where $K = 98$, $r = 0.06$, $\sigma = 0.0$, $C = 0.42$, $G = 4.37$, $M = 191.2$, $Y = 1.0102$. Exact value $\approx 9.2185$
### American Price-Convergence - CGMY

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<td>5.12880</td>
<td>0.0194</td>
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<td>320</td>
<td>5.14404</td>
<td>0.0042</td>
</tr>
<tr>
<td>640</td>
<td>5.14765</td>
<td>0.0006</td>
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</table>

Table 7: Convergence comparison of uniform and stretched grids for an American Put under CGMY. Prices at \( S = 98 \), where \( K = 98 \), \( T = 0.25 \), \( r = 0.06 \), \( \sigma = 0.0 \), \( C = 0.42 \), \( G = 4.37 \), \( M = 191.2 \), \( Y = 1.0102 \). Limiting value \( \approx 5.1485 \)

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Asian Options

Define the continuously sampled average $A_t$ price as

$$A_t = \frac{1}{t} \int_0^t S_\tau d\tau$$

where $A_0 = S_0$, and which evolves according to

$$dA_t = \frac{1}{t}(S_t - A_t)dt$$

where as before

$$dS_t/S_t = (\mu - \lambda \kappa)dt + \sigma dW_t + dJ_t$$

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Asian Pricing PDE

With no jumps, the PDE for the price $V(A, S, t)$ follows an “ultra-parabolic equation”

$$-\frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{1}{t} (S - A) \frac{\partial V}{\partial A} - rV$$

with a final condition (payoff) at time $t = T$,

$$V(S, A, T) = g(S, A, T)$$

There are transformations that reduce the dimensionality but this is not always possible. In this case the pde can be solved efficiently using finite differences combined with a semi-Lagrange integration method.
Semi-Lagrangian approach

The Lagrangian derivative $\frac{dV}{dt}$ along any path in the $A - t$ plane is given by

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial A} \frac{dA}{dt}.$$ 

The Asian pde then simplifies to

$$-\frac{dV}{dt} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = \mathcal{L}V$$

along paths $\mathcal{P}(A, t; S)$ such that

$$\frac{dA}{dt} = \frac{1}{t}(S - A).$$
**Discretisation**

Take the integral of the PDE along the path $\mathcal{P}_k^m(A, t; S_j) \equiv (\tilde{A}_k, t^m) \sim (A_k, t^{m-1})$, i.e.

$$V(S_j, A_k, t^{m-1}) - V(S_j, \tilde{A}_k, t^m) = \int_{\mathcal{P}_k^m(A, t; S_j)} \mathcal{L}_S V(S, A, t) dt.$$ 

Finite difference prices $\{V_{j,k}^m\}$ on the mesh $\{S_0, \ldots, S_N\} \otimes \{A_0, \ldots, A_N\}$ can be derived using theta method quadrature and $\mathcal{L}_{\Delta S}$, the $O(h^2)$ approximation to $\mathcal{L}$

$$V_{j,k}^{m-1} = \tilde{V}_{j,k}^m + \Delta t \left( \theta \mathcal{L}_{\Delta S}(V_{j,k}^{m-1}) + (1 - \theta) \tilde{\mathcal{L}}_{\Delta S}(V_{j,k}^m) \right)$$

$\tilde{V}_{j,k}^m$ is the mesh price interpolated to $\tilde{A}_k$. $V_{j,k}^m \approx V(S_j, A_k, t^m)$ to $O(\Delta t^2) + O(h^2)$ for $\theta = 0.5$ and the equations are unconditionally stable for $\theta \geq 0.5$. 

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Determining $\tilde{A}_k$

Integrating $(\cdot)$ with respect to time along the path $\mathcal{P}^m_k(A, t; S)$ gives

$$\int_{A_k}^{\tilde{A}_k} \frac{1}{S_j - A} dA = \int_{t^{m-1}}^{t^m} \frac{1}{t} dt$$

$$\implies \ln \left[ \frac{(S_j - A_k)}{(S_j - \tilde{A}_k)} \right] = \ln \left[ \frac{t^m}{t^{m-1}} \right]$$

thus

$$\tilde{A}_k = S_j - \left( \frac{t^{m-1}}{t^m} \right) (S_j - A_k) = \alpha A_k + (1 - \alpha) S_j, \text{ where } \alpha = \frac{t^{m-1}}{t^m}.$$
Boundary condition at \( S = 0 \)

If at some time \( t^* \) the asset price \( S_{t^*} = 0 \) then it remains zero; hence the final average price is known

\[
A_T = \frac{1}{T} \int_0^T S_t \, dt = \frac{1}{T} \int_0^{t^*} S_t \, dt = \frac{t^*}{T} A_{t^*}
\]

and hence the payoff, \( \forall A \geq 0 \) is also known, e.g. for an average rate put

\[
V(0, A, t) = e^{-r(t-T)} \max(K - \frac{t}{T} A, 0)
\]

Alternatively use the exact solution of the pde for \( S = 0 \) (it becomes linear hyperbolic).
Procedure for S-L Integration

- The solution is carried out backwards in time, so beginning with values at $T$, values at $t^{m-1}$ are obtained, and so on. A set of tri-diagonal equations are solved at each timestep (no early exercise).

- Prices and difference approximations at trajectory endpoints are obtained by cubic-spline interpolation in the A-direction.

- If the option has early exercise then the A-parameterised problems becomes parameterised LCP’s and can be solved using PSOR.
Price Evolution of a Fixed-strike Asian call
Figure 3: At $t = \text{expiry}$ (6 months).
Figure 4: At $t = 4$ months.
Figure 5: At $t = 2$ months.
Figure 6: At $t = 0$. 

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# Results with early-exercise

Table 8: Semi-Lagrangian (S-L) convergence for Fixed Strike Put Asians with early exercise, where $r = 0.1$, quoted at strike $K = 100$. $nt = 40$, for all maturities. $\lambda = 5$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$T$</th>
<th>Calculated by S-L</th>
<th>(BP, 1996)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>20\times20</td>
<td>40\times40</td>
</tr>
<tr>
<td>0.20</td>
<td>0.25</td>
<td>1.721</td>
<td>1.959</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>2.286</td>
<td>2.621</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>2.981</td>
<td>3.243</td>
</tr>
<tr>
<td>0.40</td>
<td>0.25</td>
<td>4.360</td>
<td>4.619</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>6.084</td>
<td>6.128</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>7.850</td>
<td>7.862</td>
</tr>
</tbody>
</table>
Fixed-strike Asian Put with early exercise
Figure 7: At $t = \text{expiry (3 months)}$. 

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Figure 8: At $t = 2$ months.
Figure 9: At $t = 1$ month.
Figure 10: At $t = 0$. 
Asian Pricing PIDE

With jumps, the price $V(A, S, t)$ now follows an “ultra-parabolic PIDE”

$$- \frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{1}{t}(S - A) \frac{\partial V}{\partial A} - rV$$

$$+ \int_{-\infty}^{\infty} (V(S e^x) - V(S) - S(e^x - 1)\frac{\partial V}{\partial S}) \nu(x) dx$$
This simplifies as before with semi-Lagrange integration to

\[-\frac{dV}{dt} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \int_{-\infty}^{\infty} (V(Se^x) - V(S) - S(e^x - 1) \frac{\partial V}{\partial S}) \nu(x) dx\]

\[= \mathcal{L}V + \mathcal{I}(V)\]

along paths \(\mathcal{P}(A, t; S)\) such that

\[\frac{dA}{dt} = \frac{1}{t}(S - A)\].

This structure is again an A-parameterised version of the single factor jump diffusion problem, so the same finite difference approach can be used for a fixed set of average price values.
Discretisation

Take the integral of the PIDE along the path $\mathcal{P}_m^k(A, t; S_j) \equiv (\tilde{A}_k, t^m) \sim (A_k, t^{m-1})$, i.e.

$$V(S_j, A_k, t^{m-1}) - V(S_j, \tilde{A}_k, t^m) = \int_{\mathcal{P}_m^k(A, t; S_j)} (\mathcal{L}_S V(S, A, t) + \mathcal{I}(V(S, A, t))) \, dt.$$ 

Finite difference prices $\{V_{m,j,k}\}$ on the mesh $\{S_0, \ldots S_N\} \otimes \{A_0, \ldots A_N\}$ can be derived using theta method quadrature and $\mathcal{L}_\Delta S$, the $O(h^2)$ approximation to $\mathcal{L}$

$$V_{j,k}^{m-1} = \tilde{V}_{j,k}^m + \theta \Delta t \left( \mathcal{L}_\Delta S (V_{j,k}^{m-1}) + \mathcal{I}_j (V^{m-1}) \right) + (1-\theta) \Delta t \left( \tilde{\mathcal{L}}_\Delta S (V_{j,k}^m) + \tilde{\mathcal{I}}_j (V^m) \right)$$

$\tilde{V}_{j,k}^m$ is the mesh price interpolated to $\tilde{A}_k$. $V_{j,k}^m \approx V(S_j, A_k, t^m)$ to $O(\Delta t^2) + O(h^2)$ for $\theta = 0.5$ and the equations are unconditionally stable for $\theta \geq 0.5$. 

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Comments

• Mesh based quadrature with coordinate stretching leads to accurate finite difference approximations to the for jump diffusion option pricing PIDEs (Merton, VG, CGMY)

• Semi-Lagrange time integration simplifies an Asian pricing PIDE into a A-parameterised set of one-factor problems very similar to the above and permits the same approach to early exercise

• The S-L method is unconditionally stable when combined with implicit finite differences and can be applied to discrete averaging and to volatility surfaces.

• The combined use of S-L and mesh based quadrature with coordinate transformations can be expected to lead to flexible, efficient and accurate early-exercise Asian pricing for a jump diffusion models
• For more complex Asian options e.g. assets with stochastic volatility, see Parrott & Clarke, *Parallel Solution of American Asian Options*, Procs. of the 11th Domain Decomposition Conference, 1999.


