Constraint Attribution:
Mastering Constraints for Better Portfolio Construction

Adrian Zymolka
Axioma
azymolka@axiomainc.com

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Outline

- Motivation
- The Effect of Constraints
- Use Cases
- Proposed Methodology
- Constraint Performance Attribution
Constraints...the “fiction” and the facts

• “There is no downside to constraints, only good.”

• Constraints are important and useful
  – But they can actually hurt performance

• We will introduce an optimization-based methodology for evaluating constraints that answers:
  – How do constraints affect the optimal portfolio?
  – How do constraints “modify” the expected returns?
  – How do constraints affect portfolio performance over time?
Why do practitioners impose constraints?

- **Client mandate**
  - Risk targets: Absolute or tracking error targets
  - Compliance: No Tobacco, No “Sin” stocks, Islamic mandate

- **Investment strategy**
  - Sector/Industry exposures
  - Style factor exposures (beta neutrality, value tilt)
  - Leverage (long/short, 130/30)

- **Market**
  - Liquidity considerations (holding and trading)

- **Regulatory**
  - 5/10/40 Rule

- **MVO process**
  - Reducing instability and error maximization properties of MVO
Which constraints are used in practice?

- Application on asset level, portfolio level, or any (weighted) subset level like sectors, industries, countries, factor exposures, ...
- Flexible parameters, e.g. weights $w_i$ or active weights $w_i - b_i$ (against any benchmark)
- Multiple risk models, multiple benchmarks etc. to be used in parallel

Available constraints:

- **Holdings:**
  - Budget: $m \leq \sum w_i \leq M$
  - Holdings: $m \leq \sum \lambda_i w_i \leq M$
  - Absolute Holding: $m \leq \sum |\lambda_i w_i| \leq M$
  - Threshold Holding: $w_i = 0 \lor |w_i| \geq m$
  - Long Holding: $m \leq \sum \lambda_i w_i^+ \leq M$
  - Short Holding: $m \leq \sum \lambda_i w_i^- \leq M$
  - Long/Short Ratio: $m \leq \sum \lambda_i w_i^+ / \sum \lambda_i w_i^- \leq M$
  - Weighted Avg Holding: $m \leq \sum \lambda_i w_i / |\sum \lambda_i w_i| \leq M$
  - Model Deviation: $m \leq \sum (\lambda_i w_i)^2 \leq M$
  - Issuer Holding 5/10/40 type limit

- **Risk:**
  - Risk: $\sqrt{w^T Q w} \leq M$
  - Rel. MCTR: $w^T \Lambda Q w / w^T Q w \leq M$

- **Names (counts):**
  - Names: $m \leq |\{i: w_i \neq 0\}| \leq M$
  - Long Names: $m \leq |\{i: w_i^+ \neq 0\}| \leq M$
  - Short Names: $m \leq |\{i: w_i^- \neq 0\}| \leq M$
  - Trades: $m \leq |\{i: t_i \neq 0\}| \leq M$
  - Buy Trades: $m \leq |\{i: t_i^+ \neq 0\}| \leq M$
  - Sell Trades: $m \leq |\{i: t_i^- \neq 0\}| \leq M$
Which constraints are used in practice?

- Available constraints (continued):

  - **Exposures**: (on exposures $e_i = w_i + \sum_c \lambda_{c,i} w_c$ for composites $c$)
    - Exposures: $m \leq \sum \lambda_i e_i \leq M$
    - Absolute Exposure: $m \leq \sum |\lambda_i e_i| \leq M$
    - Long Exposure: $m \leq \sum \lambda_i e_i^+ \leq M$
    - Short Exposure: $m \leq \sum \lambda_i e_i^- \leq M$
    - Issuer Exposure: $5/10/40$ type limit (on $e_i$'s)

  - **Robust**:
    - Robust Linear: $\sum \alpha_i w_i - k\|\Omega^{1/2}w\| \leq M$
    - Probabilistic Linear: $P(\alpha^T w \geq R - d\sigma) \geq \eta$

  - **Trading**: (on trades $t_i = w_i - h_i$)
    - Trade Size: $m \leq \sum \lambda_i t_i \leq M$
    - Buy Size: $m \leq \sum \lambda_i t_i^+ \leq M$
    - Sell Size: $m \leq \sum \lambda_i t_i^- \leq M$
    - Threshold Trade: $t_i = 0 \lor |t_i| \geq m$
    - Turnover: $\sum |t_i| \leq M$
    - Short Sell Cost: $\sum \lambda_i \max(0, t_i^- - h_i^+ \leq M$
    - Transaction Cost: $\sum \text{pwlinear}(t_i) \leq M$

  - **Tax**: (on trades $t_i = w_i - h_i$)
    - Tax Liabilities: $\sum \text{stl}(t_i) + \text{l tl}(t_i) \leq M$
    - Net Tax Gains: $m \leq \sum \text{stng}(t_i) + \text{l tng}(t_i) \leq M$
    - Net Tax Losses: $m \leq \sum \text{stnl}(t_i) + \text{l tnl}(t_i) \leq M$
    - Gross Tax Gains: $m \leq \sum \text{stgg}(t_i) + \text{l tgg}(t_i) \leq M$
    - Gross Tax Losses: $m \leq \sum \text{stgl}(t_i) + \text{ltgl}(t_i) \leq M$
    - Almost Long Term Gains: $t_i = 0$ if $\text{ltgp}(i) - \text{hold}(i) \leq d$
    - Min Holding Period: $t_i = 0$ if $\text{hold}(i) \leq d$
Which objectives are used in practice?

- Weighted sum of any set of individual objective terms: \( \lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_n T_n \)
- Example Utility function: \( \text{Max} \ 1 \cdot \alpha^T w + (-\lambda) \cdot w^T Q w \)
- Available objective terms \( T_i \):
  - **Tilts**:
    - Net Tilt: \( \sum a_i w_i \)
    - Tilt Long: \( \sum a_i w_i^+ \)
    - Tilt Short: \( \sum a_i w_i^- \)
  - **Returns**:
    - MVO (Exp. Return): \( \sum \alpha_i w_i \)
    - Robust Return: \( \sum \alpha_i w_i - k\|\Omega^{1/2} w\| \)
  - **Risk**:
    - Std.dev. (Risk): \( \sqrt{w^T Q w} \)
    - Variance: \( w^T Q w \)
  - **Transaction costs**:
    - Fixed Charge Buy Cost: \( \sum (c_i \text{ if } t_i^+ > 0) \)
    - Fixed Charge Sell Cost: \( \sum (c_i \text{ if } t_i^- > 0) \)
    - Fixed Charge Transaction Cost: \( \sum (c_i \text{ if } t_i > 0) \)
    - Linear Buy Cost: \( \sum c_i t_i^+ \)
    - Linear Sell Cost: \( \sum c_i t_i^- \)
    - Linear Transaction Cost: \( \sum c_i t_i \)
    - Market Impact (\( p \in \{2, \frac{3}{2}, \frac{5}{3}\} \)): \( \sum c_i t_i^p \)
    - Short Sell Cost: \( \sum c_i \max(0, t_i^- - h_i^+) \)
    - Transaction Cost: \( \sum \text{pwlinear}(t_i) \)
  - **Tax**:
    - Tax Gains: \( \sum \text{stg}(t_i) + \text{lgt}(t_i) \)
    - Tax Liabilities: \( \sum \text{stl}(t_i) + \text{ltl}(t_i) \)
How do constraints affect the optimized portfolio?

Market Data

Constraints
- Strategy
- Compliance
- Liquidity
- Regulatory

Current Portfolio

Risk Model (Q)

Expected Returns (Alphas)

Portfolio Optimizer

Maximize Utility
(Expected Return – Risk)
Subject to:
Constraints

• Unrealized Alpha
• Opportunity Cost
• Implementation Inefficiency

Optimized Portfolio (w)
The Effect of Constraints
Previous work

• Transfer Coefficient


• Effect of Constraints (Shadow Prices)


Transfer coefficient

- Measures **how well information is transferred** from the expected returns to the portfolio

- The **correlation** between “risk-adjusted expected returns” and “risk-adjusted holdings”

- Unconstrained portfolio has TC of 1.0, in general: \( IR = (TC) \ (IC) \ N^{\frac{1}{2}} \)

- Gives **aggregate effect** of ALL constraints—does NOT show how individual constraints affect the optimal portfolio

- Iteratively look at transfer coefficient with different combinations of constraints:
  - Litterman – “Are Constraints Eating Your Alpha?”
  - Clarke, de Silva, Sapra – “Toward More Information-Efficient Portfolios”
Shadow prices
(dual prices, Lagrange multipliers)

• The *shadow price* of a constraint is its associated dual variable value, or its dual Lagrange multiplier in the optimal solution.

• The shadow price of a constraint can be interpreted as the *marginal change in obj value when relaxing the constraint* by one unit.

• Example:
  – Objective is to maximize expected return: Optimum is 5%
  – Risk limit of 3% has a shadow price of 0.5 at optimum
  – For $\varepsilon$ increase in risk limit of 3%, the expected return “will change” by $0.5\varepsilon$

• Important:
  – Shadow price is only a *marginal value*
  – If the constraint limit is changed, the objective may not change exactly as predicted

• Shadow prices are provided as an *optimization by-product* of the solution of any *continuous convex* optimization problem.
Simple example

- **Objective:**
  - Maximize Utility = Expected Return – Risk Aversion * Active Variance

- **Constraints:**
  - Budget ($100,000, fully invested)
  - No asset shorted more than 5% (short limit)
  - IBM position less than 50% (long limit)

<table>
<thead>
<tr>
<th>Assumptions</th>
<th>DELL</th>
<th>IBM</th>
<th>MSFT</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha (%)</td>
<td>-5</td>
<td>7</td>
<td>4</td>
<td>-2</td>
</tr>
<tr>
<td>Benchmark Holding(%)</td>
<td>30</td>
<td>23</td>
<td>12</td>
<td>35</td>
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</table>

<table>
<thead>
<tr>
<th>Solution</th>
<th>DELL</th>
<th>IBM</th>
<th>MSFT</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained (%)</td>
<td>-230.39</td>
<td>200.45</td>
<td>101.57</td>
<td>-74.74</td>
</tr>
<tr>
<td>Constrained (%)</td>
<td>-5.00</td>
<td>50.00</td>
<td>60.00</td>
<td>-5.00</td>
</tr>
</tbody>
</table>
Shadow prices

- Shadow price for short limit (5%) on DELL is 0.06
- Change short limit from 5% to 5.01% ($10 = 0.01% * $100,000)
- New objective value: $5,290 (was $5,289.4)
- Difference: (change in short limit) * shadow price = $10 * 0.06

<table>
<thead>
<tr>
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<th>ORCL</th>
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<td>50.00</td>
<td>60.00</td>
<td>-5.00</td>
</tr>
</tbody>
</table>
Using shadow prices

- Use the shadow prices combined with partial derivatives of constraint and objective to generate a “decomposition” of the optimal solution (Grinold, Scherer & Xu)

- Optimality conditions give rise to three different ways to decompose the optimal solution:
  
  - **Implied Alpha Decomposition**
    attributes the difference between alpha and implied alpha to the constraints and objective terms
  
  - **(Optimal) Holdings Decomposition**
    attributes the difference between optimal and MV holdings, i.e., between holdings in constrained and unconstrained portfolios to constraints and objective terms
  
  - **Expected Return Decomposition**
    attributes the difference between optimal and MV expected return, i.e., expected return of constrained and unconstrained portfolio, to constraints and objective terms
Underlying optimization principles

\[ \alpha^T w - \lambda w^T Q w \]

Subject to
\[ e^T w = 1 \quad (\pi) \]
\[ w \geq 0 \quad (\mu) \]

\[ \alpha^* = 2\lambda Q w^* = \alpha - \pi e + \mu \]

\[ w^* = \frac{1}{2\lambda} Q^{-1} \alpha - \frac{1}{2\lambda} \pi Q^{-1} e + \frac{1}{2\lambda} Q^{-1} \mu \]

\[ \alpha^T w^* = \frac{1}{2\lambda} \alpha^T Q^{-1} \alpha - \frac{1}{2\lambda} \alpha^T \pi Q^{-1} e + \frac{1}{2\lambda} \alpha^T Q^{-1} \mu \]

Optimality Conditions
\[ \alpha - 2\lambda Q w - \pi e + \mu = 0 \]
\[ \mu_i w_i = 0 \]
\[ \mu \geq 0 \]

Implied Alpha Decomposition
Holdings Decomposition
Expected Return Decomposition
Implied alpha vs. holdings decomposition

- **Implied alphas** associated with a portfolio $w^*$ are the expected returns that make $w^*$ maximize the unconstrained problem.

- **Implied alpha decomposition** can be used to determine the effect of constraints on optimization in more intuitive way.

- The effect of the individual constraints on the difference between alphas and implied alphas is independent of the covariance matrix:
  - For example, constraint on a single asset can only affect implied alpha of that asset (but can affect holdings of all assets):
    \[ \alpha^* = \alpha - \pi e + \mu \]

- In contrast, **holdings decomposition** is largely impacted by covariance matrix:
  \[ w^* = \frac{1}{2\lambda} Q^{-1} \alpha - \frac{1}{2\lambda} \pi Q^{-1} e + \frac{1}{2\lambda} Q^{-1} \mu \]
Utility Decomposition (Scherer and Xu)

\[ \alpha^* = \alpha - \pi e + \mu \]

\[ U^* = \alpha^T w^* - \lambda (w^*)^T Q w^* \]

\[ U^{MV} = \alpha^T w^{MV} - \lambda (w^{MV})^T Q w^{MV} \]

\[ U^{MV} - U^* = \frac{1}{2} \left( \pi e^T (w^{MV} - w^*) - \mu^T (w^{MV} - w^*) \right) \]

Utility Difference

Attributed to budget

Attributed to Long Only
Use Cases
**Implied alpha decomposition**

\[ \alpha^* = \alpha - \pi \mathbf{e} + \mu \]

Demonstrates how each constraint modifies user alpha to obtain implied alpha.

<table>
<thead>
<tr>
<th></th>
<th>DELL</th>
<th>IBM</th>
<th>MSFT</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Alpha</strong></td>
<td>-5.00</td>
<td>7.00</td>
<td>4.00</td>
<td>-2.00</td>
</tr>
<tr>
<td><strong>Budget</strong></td>
<td>-1.67</td>
<td>-1.67</td>
<td>-1.67</td>
<td>-1.67</td>
</tr>
<tr>
<td><strong>Short Limit</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DELL</td>
<td>6.07</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>IBM</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>MSFT</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>ORCL</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>2.90</td>
</tr>
<tr>
<td><strong>Long Limit</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IBM</td>
<td>0.00</td>
<td>-4.28</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Implied Alpha (risk term)**

<table>
<thead>
<tr>
<th></th>
<th>DELL</th>
<th>IBM</th>
<th>MSFT</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.60</td>
<td>1.05</td>
<td>2.33</td>
<td>-0.77</td>
</tr>
</tbody>
</table>

Single asset constraint only affects implied alpha of that asset.

This provides a view that is uncomplicated by the correlation structure of variance term.

Implied Alpha: alphas that would result if optimal portfolio were unconstrained.
### Holdings decomposition

Associates a portfolio with each constraint and objective term

<table>
<thead>
<tr>
<th></th>
<th>DELL</th>
<th>IBM</th>
<th>MSFT</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MV (Alpha)</strong></td>
<td>-230.39</td>
<td>200.45</td>
<td>101.57</td>
<td>-74.74</td>
</tr>
<tr>
<td><strong>Budget</strong></td>
<td>-80.54</td>
<td>-39.38</td>
<td>-29.70</td>
<td>-81.78</td>
</tr>
<tr>
<td><strong>DELL</strong></td>
<td>304.66</td>
<td>-0.71</td>
<td>-12.19</td>
<td>1.62</td>
</tr>
<tr>
<td><strong>IBM</strong></td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td><strong>MSFT</strong></td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td><strong>ORCL</strong></td>
<td>0.77</td>
<td>-3.26</td>
<td>-0.30</td>
<td>145.08</td>
</tr>
<tr>
<td><strong>IBM</strong></td>
<td>0.50</td>
<td>-107.09</td>
<td>0.61</td>
<td>4.82</td>
</tr>
<tr>
<td><strong>Optimal Holdings</strong></td>
<td>-5.00</td>
<td>50.00</td>
<td>60.00</td>
<td>-5.00</td>
</tr>
</tbody>
</table>

Portfolio from Alpha term is the optimal unconstrained portfolio aka **Mean-Variance (MV)** portfolio.

Portfolio for Dell Short Limit has holdings in all assets -> due to asset correlations.

Optimal holdings (%) are sum of all constraint and objective portfolios.

\[
    w^* = \frac{1}{2\lambda} Q^{-1} \alpha - \frac{1}{2\lambda} \pi Q^{-1} e + \frac{1}{2\lambda} Q^{-1} \mu
\]
### Expected return decomposition

\[ \alpha^T w^* = \frac{1}{2\lambda} \alpha^T Q^{-1} \alpha - \frac{1}{2\lambda} \alpha^T \pi Q^{-1} e + \frac{1}{2\lambda} \alpha^T Q^{-1} \mu \]

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>MV (alpha term)</td>
<td>31.11</td>
</tr>
<tr>
<td>Budget</td>
<td>1.72</td>
</tr>
<tr>
<td>DELL</td>
<td>-15.80</td>
</tr>
<tr>
<td>IBM</td>
<td>0.00</td>
</tr>
<tr>
<td>MSFT</td>
<td>0.00</td>
</tr>
<tr>
<td>ORCL</td>
<td>-3.18</td>
</tr>
<tr>
<td>IBM (Long Limit)</td>
<td>-7.59</td>
</tr>
<tr>
<td>Expected Return</td>
<td>6.25</td>
</tr>
</tbody>
</table>

For each constraint, we compute the expected return of its attributed portfolio.

Expected return of optimal portfolio is the sum of the expected return of the attributed portfolios.
Utility Decomposition

$$U^{MV} - U^* = \frac{1}{2} \pi e^T (w^{MV} - w^*) - \mu^T (w^{MV} - w^*)$$

<table>
<thead>
<tr>
<th>Budget</th>
<th>% of difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>DELL</td>
<td>66.99</td>
</tr>
<tr>
<td>IBM</td>
<td>0.00</td>
</tr>
<tr>
<td>MSFT</td>
<td>0.00</td>
</tr>
<tr>
<td>ORCL</td>
<td>9.90</td>
</tr>
</tbody>
</table>

Attribute utility difference between optimized constrained portfolio and MV portfolio to constraints.

Main contributors to loss of utility:

IBM 31.52

100%
Realistic use case

- Universe/Benchmark: S&P 500

- $10 million in cash

- **Objective:** expected return – risk aversion * active variance

- **Constraints:**
  - Long Only
  - Fully invested (Budget)
  - Max 2% active position (Asset Bounds)
  - Max 2% active exposure to industries (Industry)
  - Max 2% active exposure to styles (Style)

- **MV Portfolio:** Optimal portfolio when optimizing without constraints.
### Implied alpha decomposition

<table>
<thead>
<tr>
<th></th>
<th>Asset-1</th>
<th>Asset-2</th>
<th>Asset-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>-6.05</td>
<td>5.69</td>
<td>0.36</td>
</tr>
<tr>
<td>Long Only</td>
<td>6.58</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Budget</td>
<td>-3.66</td>
<td>-3.66</td>
<td>-3.66</td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>0.00</td>
<td>-1.24</td>
<td>0.58</td>
</tr>
<tr>
<td>Industry</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Style</td>
<td>3.09</td>
<td>-0.74</td>
<td>2.68</td>
</tr>
<tr>
<td>Implied Alpha</td>
<td>-0.05</td>
<td>0.04</td>
<td>-0.04</td>
</tr>
</tbody>
</table>

- **Asset-1 negative alpha** increased by Long only constraint.
- **Asset-2 positive alpha** decreased by upper limit on long position.
- **Style factors** have significant impact on Asset-1 and Asset-3.
### Holdings decomposition

<table>
<thead>
<tr>
<th></th>
<th>Asset-1</th>
<th>Asset-2</th>
<th>Asset-3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Alpha (MV)</strong></td>
<td>-360%</td>
<td>453%</td>
<td>-100%</td>
</tr>
<tr>
<td><strong>Long</strong></td>
<td>321%</td>
<td>344%</td>
<td>-22%</td>
</tr>
<tr>
<td><strong>Budget</strong></td>
<td>16%</td>
<td>12%</td>
<td>-7%</td>
</tr>
<tr>
<td><strong>Asset Bounds</strong></td>
<td>9%</td>
<td>-114%</td>
<td>115%</td>
</tr>
<tr>
<td><strong>Industry</strong></td>
<td>-9%</td>
<td>-12%</td>
<td>-75%</td>
</tr>
<tr>
<td><strong>Style</strong></td>
<td>23%</td>
<td>6%</td>
<td>88%</td>
</tr>
<tr>
<td><strong>Optimal Holding</strong></td>
<td>0%</td>
<td>2%</td>
<td>0%</td>
</tr>
</tbody>
</table>

- **Alpha portfolio wants to short Asset-1 but it is adjusted by Long Only constraint.**
- **Alpha portfolio would like to use leverage to take long position in Asset-2 but it is adjusted by Long Only constraint that prevents leverage.**
- **Alpha portfolio wants to short Asset-3, but is prevented mostly by the Asset Bound and Style Constraints.**
Expected return decomposition

Shows expected return of each attributed portfolio

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Expected Return (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha (MV)</td>
<td>596.32%</td>
</tr>
<tr>
<td>Long Only</td>
<td>-502.27%</td>
</tr>
<tr>
<td>Budget</td>
<td>-18.99%</td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>-35.04%</td>
</tr>
<tr>
<td>Industry</td>
<td>-18.89%</td>
</tr>
<tr>
<td>Style</td>
<td>-16.57%</td>
</tr>
<tr>
<td>Optimal Portfolio</td>
<td>4.57%</td>
</tr>
</tbody>
</table>

Return if removed

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Return if removed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha (MV)</td>
<td>17.39%</td>
</tr>
<tr>
<td>Long Only</td>
<td>4.85%</td>
</tr>
<tr>
<td>Budget</td>
<td>4.81%</td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>4.87%</td>
</tr>
<tr>
<td>Industry</td>
<td>5.49%</td>
</tr>
</tbody>
</table>

Attributed return is **NOT** amount return will increase if constraint is removed, but rather which constraint will have **biggest impact if relaxed** (sensitivity)

If we remove the constraint and resolve, we confirm that the optimal expected return increases significantly
## Utility Decomposition

<table>
<thead>
<tr>
<th>Component</th>
<th>% of difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long Only</td>
<td>84.97</td>
</tr>
<tr>
<td>Budget</td>
<td>3.22</td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>5.87</td>
</tr>
<tr>
<td>Industry</td>
<td>3.14</td>
</tr>
<tr>
<td>Style</td>
<td>2.79</td>
</tr>
</tbody>
</table>

100%

Long Only constraint main contributor to loss in expected utility
Other ways to use decompositions

Constraint attributions can also be evaluated along various dimensions or measures, e.g.

What types of assets does a constraint impact the most?

Example: Long Only impact on asset sizes

- Russell 1000 universe, start from cash

- **Objective**: Expected Return

- **Constraints**:
  - Active Risk limit (1.5%)
  - Fully invested
  - Long Only
Impact of Long Only constraint

Long Only effect on implied alpha

Effect is generally smaller on large cap assets
Another example

**Example**: Trade Limit impact on asset sizes

- Russell 1000 Universe, starting account obtained with another (similar) strategy

**Objective**: Expected Return

**Constraints**:
- Active Risk at most 3%
- Fully invested
- Long Only
- Buy or Sell no more than 3% of ADV
Impact of Trade constraint

Effect is generally smaller on large cap assets.
Proposed Methodology
Our methodology

- Past work on shadow prices is theoretically sound but ignores implementation challenges

- How do we deal with realistic problems?

**Translating business constraints into optimization constraints**
- One business constraint may translate into more than one “math” constraint
- “Modeling tricks” to simplify the problem entail (linear) transformations that have an impact on constraints

**Convex (but non-differentiable)**
- Non-differentiable holding constraints: long, short, absolute value
- Non-differentiable trading constraints: buy, sell, or turnover

> Optimization theory generalizes to handle such cases
Non-differentiable case

Maximize $\alpha^T w - \lambda w^T Q w - c^T |w - h|$

KKT Conditions:

$\alpha - 2\lambda Q w^* - s = 0$

where

$s_i \in \begin{cases} 
-c_i, & \text{if } w_i^* < h_i \\
[-c_i, c_i], & \text{if } w_i^* = h_i \\
c_i, & \text{if } w_i^* > h_i 
\end{cases}$

$c_i |w_i - h_i|$ is the subgradient of the term $c^T |w - h|$
Handling non-differentiabilities

- When many holdings are at non-differentiable points, finding the correct subgradients is nontrivial.

- We use a multi-step process:
  1. Solve the rebalancing to find an optimal portfolio
  2. For each constraint and objective, determine the required structure of the subgradient and shadow prices
  3. Solve a **Secondary Optimization Problem** to compute valid subgradients and shadow prices
  4. Invert the appropriate risk/variance term to obtain the holding decomposition

- Secondary problem is itself a nontrivial optimization problem (The SECRET Sauce)

- This approach is guaranteed to find valid first-order conditions when constraints and objectives are all convex
Decomposition with non-convex constraints

- Optimization theory breaks down, and does not support an exact decomposition with combinatorial and non-convex constraints like:
  - Max/min number of names/trades
  - Threshold Holding or Trade
  - Relative Marginal Contribution to Risk
  - Long/Short Ratio
- Must resort to approximate/heuristic decompositions
- Naïve method: Attribute as much as possible to the convex constraints/objectives, assign the rest to the non-convex constraints
- This can lead to questionable results
Our approach: closing the gap

• We use convex constraints to **tighten the problem** and to approximate non-convex constraints

• This makes **attribution more difficult**: How to divide the effect of the new constraints among the non-convex constraints?

• Our method does yield good results

  *(Another SECRET Sauce)*
Constraint Performance Attribution
How do we explain performance?

• “Traditional” performance attribution
  – Brinson-based
    • Breaks down performance by a hierarchy
  – Factor-based
    • Breaks down performance by risk/user factors

• But what if an optimizer is used to build portfolios?

Performance is not exclusively driven by Alpha (and risk), but also driven by the constraints we impose...

  – Objective function (Utility = Alpha [- Risk])
  – Constraints (strategy, compliance, liquidity,...)
What is the effect of constraints over time?

• Should not look at **realized effect** over a single period
  - Would be equivalent to looking at a one period backtest

• One can perform performance and risk attribution using constraint portfolios as attributes
  - **Performance attribution** gives contribution of constraints to realized returns
  - **Risk attribution** gives contribution of constraints to realized volatility

• Both are **consistent with standard “performance attribution”** where constraint portfolios are used as attributes in lieu of sectors, styles, etc.
Holding decomposition can be used to decompose (active) realized returns in each period into components:

\[ R_t = \sum_i R_{ti} \]

Return of portfolio \( i \) attributed to constraint \( i \) in period \( t \)

Decompose cumulative returns using standard linking techniques:

\[ \tilde{R} = \sum_i \tilde{R}_i \]

Cumulative return attributed to constraint \( i \):

\[ \tilde{R}_i = \sum_t \beta_t R_{it} \]

Portfolio cumulative (active) return
Constraint risk attribution

- \( \sigma(R_i) = \) realized volatility of returns attributed to constraint \( i \)
- \( \rho(R_i, R) = \) correlation between returns attributed to constraint \( i \) and portfolio return

Risk decomposition:

\[
\sigma(R) = \sum_i \sigma(R_i) \rho(R_i, R)
\]

Risk attributed to constraint \( i \)
## Enhanced index constraint attribution

### Long Only - 3% Tracking Error Russell 1000

<table>
<thead>
<tr>
<th></th>
<th>Annualized Return Attribution</th>
<th>Risk Attribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio</td>
<td>14.46%</td>
<td></td>
</tr>
<tr>
<td>Benchmark</td>
<td>11.35%</td>
<td></td>
</tr>
<tr>
<td>Active</td>
<td>3.11%</td>
<td>3.22%</td>
</tr>
<tr>
<td>MVO (Alpha)</td>
<td>12.35%</td>
<td>2.79%</td>
</tr>
<tr>
<td>Long Only</td>
<td>-7.28%</td>
<td>1.25%</td>
</tr>
<tr>
<td>Industry Bounds</td>
<td>-0.57%</td>
<td>-0.22%</td>
</tr>
<tr>
<td>Budget</td>
<td>-1.69%</td>
<td>-0.06%</td>
</tr>
<tr>
<td>Max Turnover</td>
<td>-0.16%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>0.25%</td>
<td>-0.54%</td>
</tr>
<tr>
<td>Other</td>
<td>0.21%</td>
<td>0.01%</td>
</tr>
</tbody>
</table>

- **Alphas**: mix of growth factor & next month’s returns
- Long Only had **most negative impact** on return
- Long Only had **positive impact** on tracking error
- Results suggest going to **130/30 strategy**
Enhanced index constraint attribution
(Continuously compounded returns)
## 130/30 constraint attribution

<table>
<thead>
<tr>
<th></th>
<th>Annualized Return Attribution</th>
<th>Risk Attribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio</td>
<td>15.21%</td>
<td></td>
</tr>
<tr>
<td>Benchmark</td>
<td>11.35%</td>
<td></td>
</tr>
<tr>
<td>Active</td>
<td>3.86%</td>
<td>2.72%</td>
</tr>
<tr>
<td>MVO (Alpha)</td>
<td>7.05%</td>
<td>3.03%</td>
</tr>
<tr>
<td>Max Long</td>
<td>-2.53%</td>
<td>-0.12%</td>
</tr>
<tr>
<td>Industry Bounds</td>
<td>-0.15%</td>
<td>-0.08%</td>
</tr>
<tr>
<td>Budget</td>
<td>0.76%</td>
<td>-0.02%</td>
</tr>
<tr>
<td>Max Turnover</td>
<td>-1.54%</td>
<td>0.08%</td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>0.17%</td>
<td>-0.14%</td>
</tr>
<tr>
<td>Other</td>
<td>0.08%</td>
<td>-0.02%</td>
</tr>
</tbody>
</table>

- Same data and base strategy as enhanced index case but now 130/30 strategy
- Active return increased from 3.11% to 3.86%
- Information ratio increased from 0.96 to 1.42
- Constraints have no significant contribution to risk
- Leverage and turnover constraints still have negative impact on return...
## 130/30 Beta-constrained attribution

<table>
<thead>
<tr>
<th>130/30 - Beta Constrained Russell 1000</th>
<th>Annualized Return Attribution</th>
<th>Risk Attribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio</td>
<td>14.03%</td>
<td></td>
</tr>
<tr>
<td>Benchmark</td>
<td>11.35%</td>
<td></td>
</tr>
<tr>
<td>Active</td>
<td>2.68% 2.74%</td>
<td></td>
</tr>
<tr>
<td>MVO (Alpha)</td>
<td>6.23% 3.34%</td>
<td></td>
</tr>
<tr>
<td>Beta</td>
<td>-1.21% 0.02%</td>
<td></td>
</tr>
<tr>
<td>Max Long</td>
<td>-1.52% -0.17%</td>
<td></td>
</tr>
<tr>
<td>Industry Bounds</td>
<td>-0.02% -0.01%</td>
<td></td>
</tr>
<tr>
<td>Budget</td>
<td>1.20% -0.03%</td>
<td></td>
</tr>
<tr>
<td>Max Turnover</td>
<td>-2.37% -0.13%</td>
<td></td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>0.09% -0.20%</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>0.25% -0.08%</td>
<td></td>
</tr>
</tbody>
</table>

Maximizes expected return subject to the same constraints as before plus Beta constraint.

Beta constrains portfolio to have beta within +/- 0.01 of benchmark beta.

Beta has significant negative contribution to return and little contribution to risk.

Alphas unknowingly have information in market timing.
### 130/30 loosely Beta-constrained attribution

<table>
<thead>
<tr>
<th>130/30 - Loosely Beta Constrained Russell 1000</th>
<th>Annualized Return Attribution</th>
<th>Risk Attribution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Portfolio</strong></td>
<td>14.64%</td>
<td></td>
</tr>
<tr>
<td><strong>Benchmark</strong></td>
<td>11.35%</td>
<td></td>
</tr>
<tr>
<td><strong>Active</strong></td>
<td>3.28% 2.75%</td>
<td></td>
</tr>
<tr>
<td><strong>MVO (Alpha)</strong></td>
<td>6.13% 3.39%</td>
<td></td>
</tr>
<tr>
<td><strong>Beta</strong></td>
<td>-0.07% 0.02%</td>
<td></td>
</tr>
<tr>
<td><strong>Max Long</strong></td>
<td>-1.40% -0.19%</td>
<td></td>
</tr>
<tr>
<td><strong>Industry Bounds</strong></td>
<td>-0.03% -0.01%</td>
<td></td>
</tr>
<tr>
<td><strong>Budget</strong></td>
<td>0.61% -0.02%</td>
<td></td>
</tr>
<tr>
<td><strong>Max Turnover</strong></td>
<td>-2.24% -0.17%</td>
<td></td>
</tr>
<tr>
<td><strong>Asset Bounds</strong></td>
<td>0.10% -0.21%</td>
<td></td>
</tr>
<tr>
<td><strong>Other</strong></td>
<td>0.20% -0.05%</td>
<td></td>
</tr>
</tbody>
</table>

- **Same data and strategy with modified Beta constraint**
- Beta now constrains portfolio to have beta within +/- 0.05 of benchmark beta
- **Annualized active return** increased from 2.68% to 3.28%
- **Information ratio** increased from 0.98 to 1.19
- Beta now has little contribution to return
### 130/30 no size constraint constraint attribution

<table>
<thead>
<tr>
<th></th>
<th>Annualized Return Attribution</th>
<th>Risk Attribution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Portfolio</strong></td>
<td>12.17%</td>
<td></td>
</tr>
<tr>
<td><strong>Benchmark</strong></td>
<td>11.35%</td>
<td></td>
</tr>
<tr>
<td><strong>Active</strong></td>
<td>0.82% 3.16%</td>
<td></td>
</tr>
<tr>
<td><strong>MVO (Alpha)</strong></td>
<td>3.31% 5.30%</td>
<td></td>
</tr>
<tr>
<td><strong>Max Long</strong></td>
<td>-2.35% -1.81%</td>
<td></td>
</tr>
<tr>
<td><strong>Industry Bounds</strong></td>
<td>-0.27% -0.16%</td>
<td></td>
</tr>
<tr>
<td><strong>Budget</strong></td>
<td>1.39% -0.11%</td>
<td></td>
</tr>
<tr>
<td><strong>Max Turnover</strong></td>
<td>-1.02% 0.07%</td>
<td></td>
</tr>
<tr>
<td><strong>Asset Bounds</strong></td>
<td>-0.11% -0.08%</td>
<td></td>
</tr>
<tr>
<td><strong>Other</strong></td>
<td>-0.09% -0.07%</td>
<td></td>
</tr>
</tbody>
</table>

Same strategy as previous 130/30 with different expected returns

Expected returns now have size noise

The alphas tilt the portfolio towards large-cap stocks over small-cap, but contain no size information.
130/30 size-constrained constraint attribution

<table>
<thead>
<tr>
<th>130/30 - Size Constrained Russell 1000</th>
<th>Annualized Return Attribution</th>
<th>Risk Attribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio</td>
<td>13.80%</td>
<td></td>
</tr>
<tr>
<td>Benchmark</td>
<td>11.35%</td>
<td></td>
</tr>
<tr>
<td>Active</td>
<td>2.45%</td>
<td>3.25%</td>
</tr>
<tr>
<td>MVO (Alpha)</td>
<td>3.15%</td>
<td>5.12%</td>
</tr>
<tr>
<td>Size</td>
<td>1.20%</td>
<td>0.15%</td>
</tr>
<tr>
<td>Max Long</td>
<td>-2.77%</td>
<td>-1.62%</td>
</tr>
<tr>
<td>Industry Bounds</td>
<td>-0.27%</td>
<td>-0.15%</td>
</tr>
<tr>
<td>Budget</td>
<td>1.15%</td>
<td>-0.10%</td>
</tr>
<tr>
<td>Max Turnover</td>
<td>0.02%</td>
<td>-0.02%</td>
</tr>
<tr>
<td>Asset Bounds</td>
<td>-0.06%</td>
<td>-0.06%</td>
</tr>
<tr>
<td>Other</td>
<td>0.02%</td>
<td>-0.07%</td>
</tr>
</tbody>
</table>

We add a size constraint

Size constraint limits size factor exposure to be within 5% of benchmark

Annualized active return increased from 0.82% to 2.45%

Information ratio increased from 0.26 to 0.75

Size constraint has a significant positive contribution to return
Conclusions

- Identifying constraints that *can* hurt performance is a crucial aspect of the portfolio management process.
- The consistent methodology enables the efficient use of constraints and the direct understanding on their implications.
- The methodology reveals how constraints affect portfolios, alphas, and how over time constraints affect performance.
- The effective application of this methodology enhances the portfolio-construction process.
Thank you for your attention!